

Interaction of a magnetic dipole with a slowly moving electrically conducting plate

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Abstract. We report an analytical investigation of the force and torque acting upon a magnetic dipole placed in the vicinity of a moving electrically conducting nonmagnetic plate. This problem is relevant to contactless electromagnetic flow measurement in metallurgy and extends previous theoretical works (Thess et al. *Phys. Rev. Lett.*, **96**(2006), 164501; *New J. Phys.* **9**(2007), 299) to the case where the orientation of the magnetic dipole relative to the plate is arbitrary. It is demonstrated that for the case of low magnetic Reynolds number the three-dimensional distributions of the induced electric potential, of the eddy currents and of the induced magnetic field can be rigorously derived. It is also shown that all components of the force and torque can be computed without any further approximation. The results of the present work serve as a benchmark problem that can be used to verify numerical simulations of more complex magnetic field distributions.

1. Introduction

When an electrically conductive macroscopic body moves in the nonuniform magnetic field \mathbf{B} created by an external source, for instance a permanent magnet, then eddy currents \mathbf{j} are induced in the body. These eddy currents create the Lorentz force $\mathbf{F} = \mathbf{j} \times \mathbf{B}$. The force is directed in the direction opposite to the direction of the body's translation. Moreover, the eddy currents induce an additional magnetic field \mathbf{b} which interacts with the permanent in such a way as to create a force and a torque acting upon the magnet. Thus, by measuring these forces and torques one can determine the parameters of the movement such as the velocity or the direction. This principle is embodied in a contactless electromagnetic flow measurement technique called Lorentz force velocimetry [8], [10], [9], [1], [2], [6], which permits flow measurement in hot and aggressive fluids like liquid aluminium or molten steel. When developing measurement systems embodying Lorentz force velocimetry, so-called Lorentz force flowmeters, it is necessary to predict the force and torque acting upon a complex-shaped magnet system by a turbulent liquid metal flowing in pipes, ducts and open channels. Such predictions are usually performed by numerically solving the full set of three-dimensional equations of magnetohydrodynamics [3]. In order to be able to assess the reliability of such simulations it is necessary to have simple models that are amenable to rigorous analytic treatment. The goal of the present paper is to formulate and solve such a model.

The model to be studied in the present work is a generalization of a previously studied problem [9] to the case when the orientation of a magnetic dipole is arbitrary. More precisely, the authors of [9] investigated the interaction between a moving plate and a magnetic dipole whose orientation is perpendicular to the surface of the plate. Here we relax the assumption about the orientation of the dipole and allow the dipole to be arbitrarily oriented. As will be shown in Section 2, this problem can still be solved exactly and all components of the torque and the force can be expressed analytically along with the electric and magnetic field. We illustrate our results in Section 3 using some representative plots of the three-dimensional structure of the eddy currents and the induced magnetic fields. In Section 4 we summarize our conclusions and indicate a possible illustrative application of the theory that would be interesting to investigate.

2. The problem and its exact solution

Consider a single magnetic dipole with dipole moment \mathbf{m} located at a distance h above a fluid or solid layer with electrical conductivity σ and thickness w as shown in Fig. 1. The dipole is assumed to be arbitrary oriented in space. For brevity, if it is not specified otherwise, we assume that the index n denotes one of the Cartesian axes, $\mathbf{m} = m\mathbf{e}_n$. The general case can be considered as a linear combination of the three generic cases, $n = x, y, z$, any general orientation a can be expressed as $a = \mathbf{m} \cdot \{\mathbf{a}^{(n)}\}^T$, where $\{\mathbf{a}^{(n)}\} = (a^{(x)}, a^{(y)}, a^{(z)})$ is the tensor containing components for each $n = x, y, z$ dipole orientation. The layer moves with an uniform velocity $\mathbf{u} = u\mathbf{e}_x$ and extends in the z

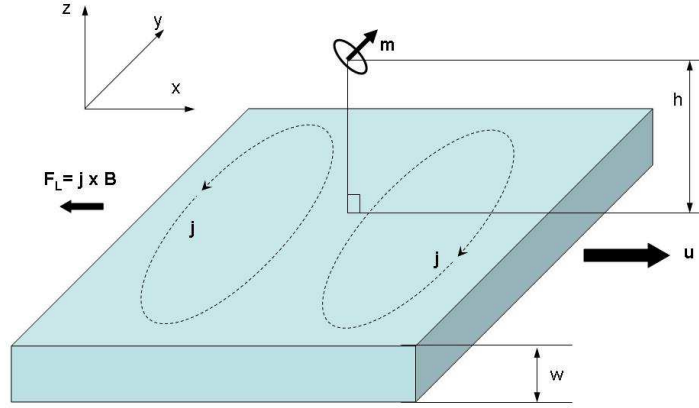


Figure 1. Sketch of the problem: A plate with thickness w moves with constant velocity u and interacts with an arbitrarily oriented magnetic dipole.

direction from z_d to z_u . The thickness of the layer is then given by $w = |z_u - z_d|$. Here, the indices "d" and "u" stand for "downward side" and "upward side", respectively. The borders of the layer in the x and y direction are supposed to be at infinity. This corresponds to the assumption that the distance h is much less than the horizontal size of the moving solid layer. We further assume that plate or the fluid moves slowly which is equivalent to the assumption that magnetic Reynolds number $Rm = \mu_0 \sigma u w$ is small. In magnetohydrodynamics (see e.g. [7], [3]) this case is referred to as the quasistatic approximation and implies that the magnetic field associated with the eddy currents is much smaller than the applied magnetic field. The assumption $Rm \ll 1$ makes the present problem amenable to rigorous analytic treatment. For instance, the magnetic Reynolds number of liquid aluminium ($\sigma \approx 3 \times 10^6$) flowing with a velocity $u = 1$ m/s in a pipe with a diameter of 0.1 m is approximately 0.3. In what follows we will use mainly Cartesian coordinates and will not apply an integration of Bessel functions in cylindrical coordinates as we did before in [9]. Our goal is to derive explicit compact algebraic expressions, which are useful for fast analytic calculations, for the primary magnetic field $\mathbf{B}(\mathbf{r})$, the electric potential $\phi(\mathbf{r})$, the eddy currents $\mathbf{j}(\mathbf{r})$ and the secondary magnetic field $\mathbf{b}(\mathbf{r})$. Then, all other quantities such as forces and torques acting upon the dipole can be readily written down.

2.1. Primary magnetic field

The first step in the solution of our problem is the specification of the magnetic field of the dipole which is referred to as the primary magnetic field. Let the magnetic dipole be at the point $\mathbf{r}_0 = (x_0, y_0, z_0)$ and we wish to calculate the magnetic field $\mathbf{B}(\mathbf{r})$ at a point $\mathbf{r} = (x, y, z)$. Before presenting formulae for the primary magnetic field, we introduce the following short notations of the vector $\mathbf{R} = \mathbf{r} - \mathbf{r}_0$:

$$\begin{aligned} \Delta x &= x - x_0, & \Delta y &= y - y_0, & \Delta z &= z - z_0, \\ \Delta r^2 &= \Delta x^2 + \Delta y^2, & R^2 &= \Delta r^2 + \Delta z^2, \end{aligned}$$

$$\frac{1}{R} = \frac{\partial}{\partial z} \left[\tanh^{-1} \frac{\Delta z}{R} \right], \quad \nabla \left[\frac{1}{R} \right] = -\frac{\mathbf{R}}{R^3} \quad (1)$$

Now, the magnetic field of a magnetic dipole can be represented as [5]:

$$\begin{aligned} \mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \nabla \times \left[\mathbf{m} \times \frac{\mathbf{R}}{R^3} \right] = \frac{\mu_0}{4\pi} \nabla \times \left[-\mathbf{m} \times \left(\nabla \frac{1}{R} \right) \right] \\ &= \frac{\mu_0}{4\pi} \nabla \times \left[\nabla \times \left(\frac{\mathbf{m}}{R} \right) - \frac{1}{R} \nabla \times \mathbf{m} \right] = \frac{\mu_0}{4\pi} \nabla \left[\mathbf{m} \cdot \nabla \left(\frac{1}{R} \right) \right], \end{aligned} \quad (2)$$

These expressions appear more complicated than the formula normally used. However, the present notation will turn out to be convenient for the further analysis and it is the key to a compact representation of all fields. Then it follows, by taking (1) and $\mathbf{m} = m\mathbf{e}_n$ into account, that the components of the primary magnetic field can be conveniently given as:

$$B_k = \mu \partial_k \partial_n \left[\frac{1}{R} \right] = \mu \partial_k \partial_n \partial_z \left[\tanh^{-1} \frac{\Delta z}{R} \right], \quad k = x, y, z, \quad (3)$$

where we introduced the abbreviation $\mu = m\mu_0/4\pi$, and $\partial_n(\cdot)$ is the partial derivative with respect to the coordinate specified by the direction \mathbf{n} of the dipole. If the dipole \mathbf{m} is arbitrary oriented in space, $\mathbf{m} = m_x\mathbf{e}_x + m_y\mathbf{e}_y + m_z\mathbf{e}_z$, then $\partial_n \equiv \mathbf{m} \cdot \nabla$. For instance, for the dipole oriented parallel to the z -axis, $\mathbf{n} = \mathbf{e}_z$ and $n = z$, and the components of the primary magnetic field are:

$$B_x = 3\mu \frac{\Delta x \Delta z}{R^5}, \quad B_y = 3\mu \frac{\Delta y \Delta z}{R^5}, \quad B_z = \mu \left[\frac{3\Delta z^2}{R^5} - \frac{1}{R^3} \right]. \quad (4)$$

2.2. Electric potential

The derivation of the electric potential is an extension of the method developed earlier in [9], and it is presented also in [6].

The electric potential ϕ and eddy currents \mathbf{j} are governed by Ohm's law for an electrically conducting material moving with velocity \mathbf{u} , which can be written as:

$$\mathbf{j} = \sigma(-\nabla\phi + \mathbf{u} \times \mathbf{B}), \quad \nabla \cdot \mathbf{j} = 0. \quad (5)$$

(The second expression is not Ohm's law and does impose the requirement that there is no source or sink of electric currents in the moving plate.) In general, the magnetic field in (5) is the sum of the primary magnetic field and the magnetic field associated with the eddy currents, referred to as the secondary magnetic field. As explained earlier, we assume that the magnetic Reynolds number $Rm = \mu_0\sigma v h$, which characterizes the ratio between the secondary and primary field, is small, so we can approximate the magnetic field in Ohm's law by the primary field given by (3).

The primary magnetic field is solenoidal, $\nabla \times \mathbf{B} = 0$, and the velocity \mathbf{u} is constant. By applying the divergence operator to Ohm's law (5) we therefore obtain the following equation for the electric potential:

$$\Delta\phi = \nabla \cdot [\mathbf{u} \times \mathbf{B}] = \mathbf{B} \cdot [\nabla \times \mathbf{u}] - \mathbf{u} \cdot [\nabla \times \mathbf{B}] = 0. \quad (6)$$

Hence, ϕ is a harmonic function obeying Laplace's equation, $\Delta\phi = 0$. Any harmonic function in three-dimensional space can be expressed through derivatives or integrals of the function $1/R$ defined in (1), where the specific expression for ϕ depends on boundary conditions. In our case, the domain is infinitively large in x and y directions, and finite in the z direction. At the upper ($z = z_u$) and lower ($z = z_d$) surfaces of the plate, the vertical electric current must be zero, which is expressed as $j_z|_{z=z_s} = 0$, and hence, $\sigma(-\partial_z\phi + uB_y)|_{z=z_s} = 0$, where the index s stands for either $s = u$ or $s = d$. Hence, we can directly express $\partial_z\phi$ by setting $j_z = 0$ everywhere in the layer:

$$\partial_z\phi = u B_y = u\mu \partial_y\partial_n\partial_z \left[\tanh^{-1} \frac{\Delta z}{R} \right]. \quad (7)$$

where the last equality directly follows from (3). Now, if we change the order of differentiation and put ∂_z as the first derivative, then the integration is trivial, and we come immediately to the final result:

$$\phi(x, y, z) = u\mu \left\{ \partial_y\partial_n \left[\tanh^{-1} \frac{\Delta z}{R} \right] + \phi_\infty(x, y) \right\}, \quad (8)$$

where the second term $\phi_\infty(x, y)$ appears as constant of the integration and so it does not depend on z . This term is determined from the additional boundary condition specifying that the electric potential vanishes when the dipole is at infinite distance from the layer. The result is:

$$\begin{aligned} \phi_\infty(x, y) &= - \lim_{z_0 \rightarrow \infty} \partial_y\partial_n \left[\tanh^{-1} \frac{\Delta z}{R} \right] \\ &= - \partial_n \left\{ \lim_{z_0 \rightarrow \infty} \partial_y \left[\tanh^{-1} \frac{\Delta z}{R} \right] \right\} \\ &= \partial_n \left\{ \operatorname{sgn}(\Delta z) \frac{\Delta y}{r} \right\} = \operatorname{sgn}(\Delta z) \partial_y\partial_n \ln r, \end{aligned} \quad (9)$$

where $\operatorname{sgn}(z) = z/|z|$ is the sign function with the property that $\operatorname{sgn}(\Delta z) = 1$ for $z > z_0$ and $\operatorname{sgn}(\Delta z) = -1$ for $z < z_0$. One can readily verify that ϕ given by (8) is the solution of (6). To do this, we notice that $\phi = \phi(1/R)$, where $1/R$ is given by (1) and $\Delta[1/R] = \Delta[\tanh^{-1}(\Delta z/R)] = 0$, hence, $\Delta\phi = 0$ too.

In Appendix A it is shown that the components of electric current can be expressed through a stream function ψ defined as $\mathbf{j} = \nabla \times (\psi \mathbf{e}_z)$. This stream function is given by:

$$\begin{aligned} \psi(x, y, z) &= -u\mu\sigma \left\{ \partial_x\partial_n \left[\tanh^{-1} \frac{\Delta z}{R} \right] + \psi_\infty(x, y) \right\}, \\ \psi_\infty(x, y) &= - \lim_{z_0 \rightarrow \infty} \partial_x\partial_n \left[\tanh^{-1} \frac{\Delta z}{R} \right] = \operatorname{sgn}(\Delta z) \partial_x\partial_n \ln r. \end{aligned} \quad (10)$$

In the short notation both ϕ and ψ can be conveniently represented by means of the differentiation of the same auxiliary function Φ as:

$$\begin{aligned} \phi^{(n)} &= u\mu \partial_y\partial_n\Phi, & \psi^{(n)} &= -u\mu\sigma \partial_x\partial_n\Phi, \\ \Phi &= \operatorname{sgn}(\Delta z) \left\{ \tanh^{-1} \frac{|\Delta z|}{R} + \ln r \right\}, \end{aligned} \quad (11)$$

where an upper index n specifies explicitly the direction of the magnetic dipole.

Because the electric currents are two-dimensional, the solution (11) describes an electric potential and current stream function for any horizontal plane of the moving layer. Explicit formulae for ψ and \mathbf{j} are given in Appendix A. These formulae can be immediately used for calculation without performing cumbersome differentiation.

2.3. Secondary magnetic field and scalar potential

The eddy currents \mathbf{j} of the moving plate induce a magnetic field \mathbf{b} which we refer to as the secondary magnetic field. When \mathbf{j} is time-independent as in our case, the secondary magnetic field $\mathbf{b}(x, y, z)$ at a point $\mathbf{r} = (x, y, z)$ is uniquely determined by Ampere's law

$$\mathbf{j}(x, y, z) = \frac{1}{\mu_0} \nabla \times \mathbf{b}(x, y, z). \quad (12)$$

On the other hand, \mathbf{j} can be determined with the aid of the stream function which has been already above calculated, Eq.(11):

$$\mathbf{j}(x, y, z) = \Theta(z) \nabla \times [\mathbf{e}_z \psi(x, y, z)].$$

Here the auxiliary step function $\Theta(z)$ is introduced in order to ensure that eddy currents outside of the plate are zero. For instance, in the case of the layer of finite thickness, the function $\Theta(z)$ is defined as:

$$\Theta(z) = \begin{cases} 1 & \text{for } z_d \leq z \leq z_u; \\ 0 & \text{for } z < z_d \text{ and } z_u < z, \end{cases}$$

where z_d and z_u are the lower and upper borders of the plate. In the general case, the secondary field can be represented completely as a sum of the stream function and a gradient of a scalar function a in the form:

$$\mathbf{b}(x, y, z) = \Theta(z) \mu_0 \psi(x, y, z) \mathbf{e}_z + \mu_0 \nabla a(x, y, z), \quad (13)$$

Function $\Theta(z)$ enforces the condition that outside the plate the secondary field \mathbf{b} is solely a gradient of the scalar potential a . This automatically ensures that $\nabla \times \mathbf{b} = 0$, and then the problem how to calculate \mathbf{b} is reduced into the problem how to calculate the scalar potential a . In order to accomplish this task one has to employ the constraint that \mathbf{b} is divergence-free. By taking the divergence of (13), the divergence-free property, $\nabla \cdot \mathbf{b} = 0$, results in the following equation:

$$\Delta a = -\Theta(z) \partial_z \psi = \Theta(z) u \mu \sigma \partial_x \partial_n \left[\frac{1}{R} \right], \quad (14)$$

where the second equality directly follows from Eq. (11). So one has to solve now equation (14) with the boundary conditions $|\nabla a| \rightarrow 0$ which describes that the secondary magnetic field vanishes at infinity.

It is possible to solve Eq. (14) in a straightforward way for any specific $n = x, y, z$. This requires explicit partial derivatives with respect to x and n and then cumbersome symbolic calculations for each separate index n by recruiting series expansion with Bessel functions and further integration. Such procedure has been performed in [9] for the

particular case $n = z$, see also [6]. However, there exists another technique to provide a general formula for arbitrary general n . For this we notice that a derivative of any function $f(R)$, which depends on R solely, with respect to $n = x, y, z$ can be taken as a derivative, with the opposite sign, with respect to $n_0 = x_0, y_0, z_0$, correspondingly, $\partial_n f(R) = -\partial_{n_0} f(R)$, where R is defined in (1). In order to use this property of R we represent the scalar potential a by means of a generating (or primitive) function A twice differentiated with respect to the coordinates of the dipole, $a = u\mu\sigma \partial_{x_0} \partial_{n_0} A$. We insert the above formula for a into (14) and obtain the simpler Poisson's equation independent of n to determine A as:

$$\Delta A = \Theta(z) \frac{1}{R}. \quad (15)$$

This equation can be readily solved without cumbersome integration and series expansion as we shall demonstrate below. Once we obtain the primitive function A , the secondary magnetic field can be represented by means of triple differentiation in the form:

$$\mathbf{b}^{(n)} = \mu_0 \mu \sigma u \nabla [\partial_{x_0} \partial_{n_0} A]. \quad (16)$$

2.4. Scalar potential and secondary magnetic field for an infinitely thin layer

Because the problem under consideration is linear, we treat the case of an infinitely thin layer separately. Once a solution for this case is found, all other cases can be presented through a superposition of thin layers.

Let \hat{z} be a location inside a thin moving layer with electric currents induced by an external magnetic dipole. Let us assume that the thickness of the layer is small compared to the distance between the dipole and the layer which is expressed by the condition $w = |z_u - z_d| \ll h$. Finally let us introduce $\hat{z} = |z_u + z_d|/2$. In this case, the function $\Theta(z)$, which is needed to set the position of the layer, must be taken in the limit $z_d \rightarrow z_u$. It gives $\Theta(z) = w \delta(z - \hat{z})$, where w is the thickness of the layer which is inserted for the proper physical dimensionality. (Function $\Theta(z)$ is dimensional in the case of the infinitely thin layer, but w will be omitted for other cases because there the proper dimensionality follows directly from an integration over the plate thickness.)

Let A_0 be the primitive function for the thin layer. Here the index 0 is inserted to distinguish the present case from other cases. As follows from Eq.(15), everywhere in the space $\Delta A_0 = 0$ except for $z = \hat{z}$. This means that A_0 is a continuous harmonic function showing a discontinuity in the first derivative with respect to z at $z = \hat{z}$. As shown in Appendix B, the solution of (15) is:

$$A_0(\hat{z}) = \frac{w}{2} \left\{ \tanh^{-1} \frac{|\hat{\Delta}z(\hat{z})|}{\hat{R}(\hat{z})} + \ln r \right\}, \quad (17)$$

$$\hat{R}(\hat{z})^2 = r^2 + |\hat{\Delta}z(\hat{z})|^2, \quad |\hat{\Delta}z(\hat{z})| = |\hat{z} - z_0| + |\hat{z} - z|.$$

Notation $|\hat{\Delta}z(\hat{z})|$ looks ugly but this is because we want to stress an analogy in between functions R and \hat{R} as explained below. In the given case of a infinitely thin layer, it

is superfluous to put argument \hat{z} in the round brackets of the function $\hat{\Delta}z$, but this argument will be necessary afterwards, when we shall extend these results to the case of a layer with finite thickness.

Note, that A_0 in (17) and Φ in (11) are almost identical, that is, $(w/2)\Phi$ corresponds to $\text{sgn}(\hat{\Delta}z)A_0$, with the mapping $R \leftrightarrow \hat{R}$ and $\Delta z \leftrightarrow \hat{\Delta}z$. Moreover, further analysis shows that this correspondence is valid not only for the entire functions A_0 and Φ , but also for all their derivatives with respect to $n = x, y, z$ and $n_0 = x_0, y_0, z_0$, that is, $(w/2)\partial_n \Phi \leftrightarrow \text{sgn}(\hat{\Delta}z)\partial_n A_0$, $\partial_n \Phi = -\partial_{n_0} \Phi$, and $\partial_n A_0 = -\partial_{n_0} A_0$. The only exception is $\partial_z A_0 = \partial_{z_0} A_0$ because $(z - z_0)$ is taken in A_0 as an absolute value. By taking this property into account and recalling that both the eddy currents \mathbf{j} and secondary magnetic field \mathbf{b} can be represented through a triple differentiation as:

$$\frac{\mathbf{j}}{\mu\sigma u} = -\nabla \times [\partial_x \partial_n \Phi] \mathbf{e}_z, \quad \frac{\mathbf{b}}{\mu_0 \mu \sigma u} = \nabla [\partial_{x_0} \partial_{n_0} A_0].$$

Therefore without computing derivatives, it is possible to build up the following correspondence between the secondary magnetic field and eddy currents:

$$b_x^{(n)} \leftrightarrow \frac{\text{sgn}(\hat{\Delta}z)}{2} j_y^{(n)}, \quad b_y^{(n)} \leftrightarrow -\frac{\text{sgn}(\hat{\Delta}z)}{2} j_x^{(n)} \quad \text{for } n = x, y \quad (18)$$

$$b_x^{(z)} \leftrightarrow -\frac{\text{sgn}(\hat{\Delta}z)}{2} j_y^{(z)}, \quad b_y^{(z)} \leftrightarrow \frac{\text{sgn}(\hat{\Delta}z)}{2} j_x^{(z)} \quad \text{for } n = z. \quad (19)$$

(The difference in sign for $n = z$ is because $\partial_z \Phi = -\partial_{z_0} \Phi$ while $\partial_z A_0 = \partial_{z_0} A_0$.) The remaining z -components of the secondary field are:

$$b_z^{(x)} \leftrightarrow \frac{\text{sgn}(\hat{\Delta}z)}{2} j_y^{(z)}, \quad b_z^{(y)} \leftrightarrow \frac{\text{sgn}(\hat{\Delta}z)}{2} j_x^{(z)},$$

$$b_z^{(z)} \leftrightarrow \frac{\text{sgn}(\hat{\Delta}z)}{2} \{j_y^{(x)} - j_x^{(y)}\}. \quad (20)$$

The last relationship for $b_z^{(z)}$ is the conjecture of the fact that the Φ is a harmonic function, i.e. $\Delta \Phi = 0$, hence, $(j_y^{(x)} - j_x^{(y)})/(\mu\sigma u) = \partial_x (\partial_x \partial_x + \partial_y \partial_y) \Phi = [-\partial_x \partial_z \partial_z \Phi]$ and $(w/2) \partial_x \partial_z \partial_z \Phi \leftrightarrow \text{sgn}(\hat{\Delta}z) \partial_x \partial_z \partial_z A_0$. According to the mapping (18-20), the tensor $\{\mathbf{b}^{(n)}\}$ of the secondary magnetic field can be now written down by means of the tensor $\{\mathbf{j}^{(n)}\}$ given in (A.10) for eddy currents. This tensor is presented in Appendix C and can be straightforwardly used in algebraic calculations in contrast to the implicit results published by Priede [6].

2.5. Secondary magnetic field for half-space and for the layer with finite thickness

Because the problem under consideration is linear, the secondary field of any moving plate can be represented by summing all the fields originating from the layers of the plate derived in the preceding section. Mathematically, this means an integration of the secondary magnetic field tensor, cf. Eq. (C.1), over the thickness of the plate. The limits of the integration in the case of half-space are z_u and $z_d \rightarrow -\infty$. Then, the integration should be carried out for the P_k coefficients only because the functions f_k and g_k do not

depend on z . By performing this procedure formally, we obtain

$$P_{\infty,k} = \int_{-\infty}^{z_u} P_{0,k}(\hat{z}) d\hat{z}, \quad (21)$$

where $P_{0,k}$ and $P_{\infty,k}$ are tensor coefficients of an infinitely thin layer and half-space, correspondingly. Explicit formulae for $P_{\infty,k}$ are presented in Appendix D.

When the plate is of finite thickness, $z_d \leq \hat{z} \leq z_u$, the coefficients P_k are obviously expressed through the coefficients for half-space in the form:

$$\begin{aligned} P_k(z_u, z_d) &= \int_{z_d}^{z_u} P_{0,k}(\hat{z}) d\hat{z} = \int_{-\infty}^{z_u} P_{0,k}(\hat{z}) d\hat{z} - \int_{-\infty}^{z_d} P_{0,k}(\hat{z}) d\hat{z} \\ &= P_{\infty,k}(z_u) - P_{\infty,k}(z_d), \end{aligned} \quad (22)$$

where $P_{\infty,k}(z_u)$ and $P_{\infty,k}(z_d)$ are computed according to (D.1) with $\hat{\Delta}z$ and \hat{R} taken at points z_u and z_d .

2.6. Magnetic potential energy, force and torque

In the general case, the potential energy U of a magnetic dipole \mathbf{m} in a magnetic field \mathbf{B} is $U = -\mathbf{m} \cdot \mathbf{B}$, see e.g. [4]. In our case, the magnetic field is the induced secondary magnetic field $\mathbf{b} = \mu_0 \mu \sigma u [\mathbf{m} \cdot \{\mathbf{b}^{(n)}\}^T \cdot \mathbf{m}]$, hence, the potential energy of the dipole is written down as:

$$\begin{aligned} U &= -\mu_0 \mu \sigma u [\mathbf{m} \cdot \{\mathbf{b}^{(n)}\}^T \cdot \mathbf{m}] . \\ &= -\mu_0 \mu \sigma u [m_x^2 b_x^{(x)} + m_y^2 b_y^{(y)} + m_z^2 b_z^{(z)} + 2 m_x m_y b_x^{(y)}] , \end{aligned} \quad (23)$$

where the last relationship is obtained by taking into account a symmetry of the secondary field tensor $b_y^{(x)} = b_x^{(y)}$, $b_y^{(z)} = -b_z^{(y)}$, and $b_x^{(z)} = -b_z^{(x)}$.

Force \mathbf{F} acting upon the dipole can be calculated as a negative gradient of the potential energy:

$$\mathbf{F} = -\nabla U, \quad F_i = \mu_0 \mu \sigma u [\mathbf{m} \cdot \{\mathbf{f}_i^{(n)}\}^T \cdot \mathbf{m}] , \quad (24)$$

where a force field tensor $\{\mathbf{f}_i^{(n)}\}$ is a partial derivative of the secondary magnetic field tensor with respect to the coordinate axis $i = x, y, z$, i.e.:

$$\{\mathbf{f}_i^{(n)}\} = \partial_i \{\mathbf{b}^{(n)}\}. \quad (25)$$

Because the tensor $\{\mathbf{b}^{(n)}\}$ has been found above, algebraic formulae for the tensor $\{\mathbf{f}_i^{(n)}\}$ can be computed as well. They are presented by Eq. (E.1-E.3). Notice, that these tensors contain 27 terms (3 times 9), while due to symmetry and algebraic simplifications they can be given by means of linear combinations of five coefficients Q_k , $k = 0, \dots, 4$.

The torque \mathbf{T} acting upon the dipole is:

$$\mathbf{T} = \mathbf{m} \times \mathbf{b} = -\mu_0 \mu \sigma u (\mathbf{m} \times [\{\mathbf{b}^{(n)}\}^T \cdot \mathbf{m}]) . \quad (26)$$

It can be computed also also by means of the tensor $\{\mathbf{b}^{(n)}\}$ presented explicitly in (C.1).

Above results for the potential energy, force and torque are of the same structure when the secondary magnetic field is induced by a discrete ensemble of magnetic dipoles

or by a continuous macroscopic magnet. (The latter can be obtained by an integration over the space occupied by the magnet.) However in our simplest case, in order to calculate the force and torque acting upon a single dipole, we just put $r = 0$ and $z = z_0$ in all the tensors found already. Mathematically, these tensors are sums of the terms, which have functions f_k and g_k as factors with coefficients P_k or Q_k . The functions f_k and g_k are equivalent to $\cos k\phi$ and $\sin k\phi$, correspondingly, here, $\tan \phi = \Delta y / \Delta x$, and, hence, f_k and g_k are zero at $r = 0$ for $k \geq 1$. Hence, at $r = 0$, all the tensor terms having factors P_k or Q_k with $k \geq 1$ must vanish due to their products with f_k and g_k . This greatly simplifies all the formulae, and we obtain finally:

$$\mathbf{b} = \mu_0 \mu \sigma u P_0|_{r=0, z=z_0} \{-m_z \mathbf{e}_x + m_x \mathbf{e}_z\}, \quad (27)$$

$$\mathbf{F} = \mu_0 \mu \sigma u Q_0|_{r=0, z=z_0} \{(3m_x^2 + m_y^2 + 4m_z^2)\mathbf{e}_x + 2m_x m_y \mathbf{e}_y\}, \quad (28)$$

$$\mathbf{T} = \mu_0 \mu \sigma u P_0|_{r=0, z=z_0} \{m_x m_y \mathbf{e}_x - (m_x^2 + m_z^2)\mathbf{e}_y + m_y m_z \mathbf{e}_z\}, \quad (29)$$

where the non-zero tensor coefficients P_0 and Q_0 at $r = 0$ and $z = z_0$ are presented in Appendix F in formulae (F.1, F.2) for a thin layer, (F.3, F.4) for a half space, and (F.5, F.6) for the layer with finite thickness.

3. Spatial structure of eddy currents and secondary magnetic field

In order to illustrate the analytical results of previous sections, we plotted eddy currents and the secondary magnetic field by using the formulae derived above, Eq. A.6) and (C.1). The eddy currents below are shown as contour lines of the eddy-current stream function ψ given by Eq. (A.6). Solid lines correspond to positive ψ and indicate counterclockwise direction of the eddy current; whereas dotted lines are for negative contour levels and imply clockwise direction of the eddy current. The secondary magnetic field is given in the form of selected three-dimensional magnetic field lines computed by using Eq. (C.1). According to the right-hand rule, the secondary magnetic field lines emanate from the solid positive loops and penetrate the negative dotted loops.

Figure 2 shows behavior at $m_y = 0$ and continuously changing m_x and m_z provided that $m_x^2 + m_y^2 + m_z^2 = 1$. This corresponds to the case when the dipole is confined to the plane spanned by the velocity vector and the direction is normal to the surface of the plate. The results of Fig. 2(a) have already been given in [9]. In this case, $m_x = m_y = 0, m_z = 1$, and there are two loops of eddy currents whose centers are arranged parallel to the direction of the plate movement. The secondary magnetic field lines form a cage. When m_x contribution increases, both loops deform and the positive loop grows in its size and shifts under the dipole as shown in Fig. 2(b). Then, the case of Fig. 2(c) corresponds to a magnetic dipole, which is oriented along the x -direction, i.e. $m_x = 1, m_y = 0, m_z = 0$, where the positive eddy current loop is located completely under the dipole, and two negative loops are at the periphery. As a result, the three-dimensional lines of the secondary magnetic field are either completely in the $y - z$ plane, if they start at $x = 0$, or the lines are closed at the peripheral negative loops if they start at $y = 0$. This behavior is emphasized by Fig. 2(d) which refers to the same

parameters as Fig. 2(c) but plotted from a different perspective. Then, Fig. 2(e) is a mirror of Fig. 2(b) with the difference that the directions of the eddy currents and of the secondary magnetic field are opposite. The same is true for Fig. 2(f) as compared to Fig. 2(a).

Figure 3 shows the behavior at $m_z = 0$ and continuously changing m_x and m_y provided that $m_x^2 + m_y^2 + m_z^2 = 1$. This corresponds to the case when the magnetic dipole is located in a plane parallel to the surface of the layer. The case of Fig. 3(a), $m_x = 1, m_y = 0, m_z = 0$ is the same as Fig. 2(c) and is repeated here in order to guide the eye. By rotating the dipole, the central loop turns together with the lines of the secondary magnetic field, Fig. 3(b). Then, when the main axis of the central loop is parallel to the diagonal of the plate, it splits into two equal positive loops, and thus one observes four alternating eddy currents loops for $m_x = 0, m_y = 1, m_z = 0$, see Fig. 3(c). The magnetic lines emanate from the positive loop and go into the neighboring negative loop. If one looks at the system from above, as in Fig. 3(d) being the same as Fig. 3(c) but differently oriented, one can see that the line $r = 0$, where the dipole is located, is the place where the magnetic field lines meet and turn out, holding the secondary field equal always zero. Then, the Fig. 3(e) is a mirror of Fig. 3(b) and Fig. 3(f) is a mirror of Fig. 3(a) with the difference that directions of eddy currents and secondary field are opposite in respect to each other.

4. Conclusion

An analytic solution is obtained for the electromagnetic interaction between an arbitrarily oriented magnetic dipole and a moving electrically conducting solid layer. The solution includes the electric potential and eddy electric currents inside the layer as well as the scalar potential for the secondary magnetic field outside the layer. The formulae obtained are useful for a Lorentz force velocimetry, because a moving solid plate can be considered as the limiting case of the mean velocity profile of a turbulent liquid metal flow when the Reynolds number (Re) tends to infinity. It has been shown in [9] that the signal of a Lorentz force velocimetry system interacting with a turbulent pipe flow converges towards the signal generated by a moving solid cylinder when $\text{Re} \rightarrow \infty$. Hence the present model provides a benchmark for numerical simulations of Lorentz force velocimetry in channel flows at high Reynolds numbers.

It should be finally mentioned that the present model could be applied in a straightforward way to describe the behavior of an educational experiment which consists of a spherical permanent magnet rolling down an inclined aluminium plate. Depending on the initial orientation of the magnetization the sphere undergoes different kinds of motion including straight motion with periodically changing velocity, wiggling motion and irregular motion. With minor changes the present model could be extended to cover this case.

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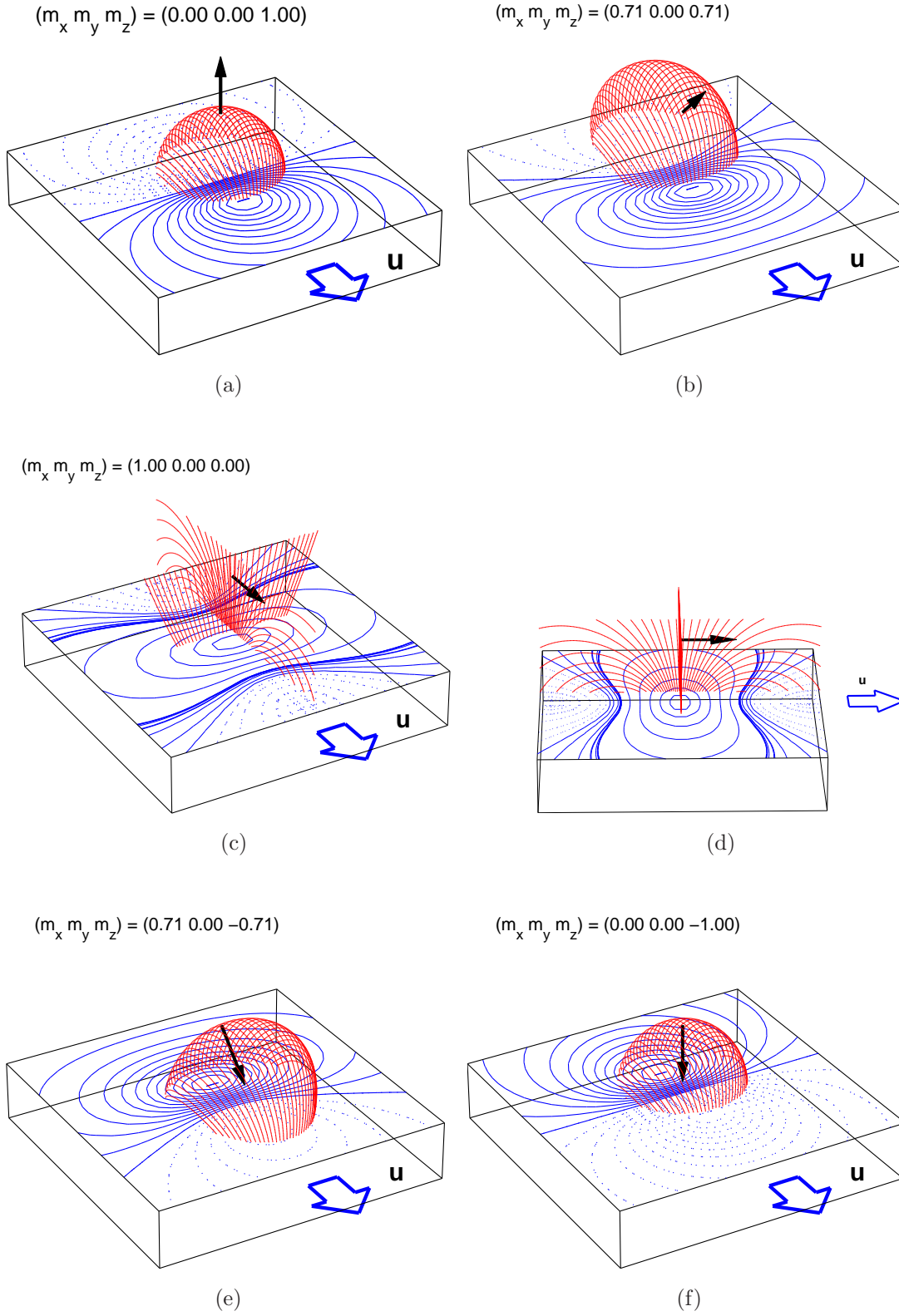
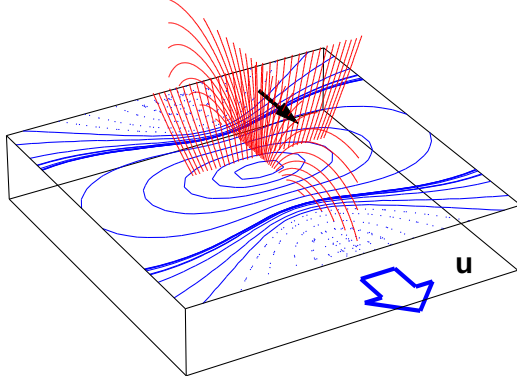


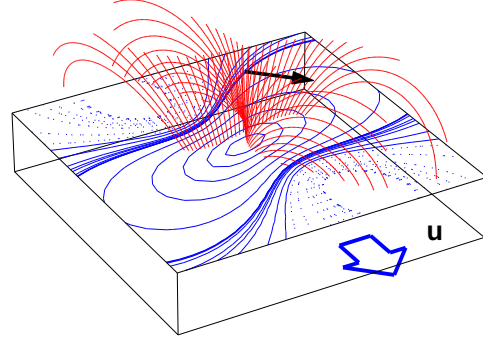
Figure 2. Eddy currents and secondary magnetic field for the case $m_y = 0$, i.e. when the magnetic dipole is located in the plane spanned by the velocity vector and the direction normal to the surface of the plate. The points where the secondary magnetic field lines emanate are selected manually inside and nearby the eddy-current loops in order to guide the eye.

$$(m_x \ m_y \ m_z) = (1.00 \ 0.00 \ 0.00)$$



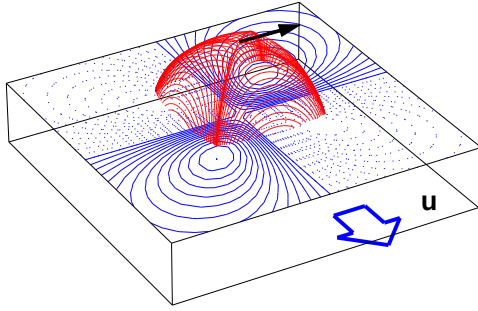
(a)

$$(m_x \ m_y \ m_z) = (0.71 \ 0.71 \ 0.00)$$

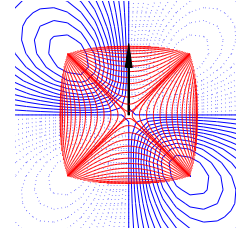


(b)

$$(m_x \ m_y \ m_z) = (0.00 \ 1.00 \ 0.00)$$

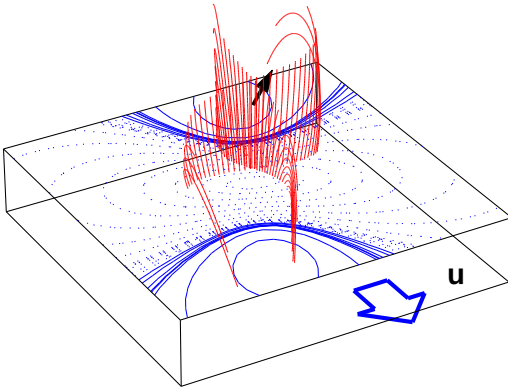


(c)



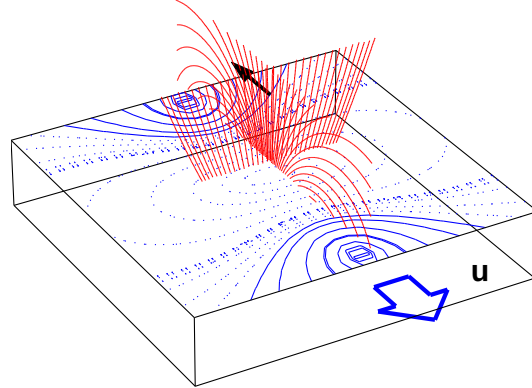
(d)

$$(m_x \ m_y \ m_z) = (-0.71 \ 0.71 \ 0.00)$$



(e)

$$(m_x \ m_y \ m_z) = (-1.00 \ 0.00 \ 0.00)$$



(f)

Figure 3. Eddy currents and secondary magnetic field for the case $m_z = 0$, i.e. when the magnetic dipole is located in a plane parallel to the surface of the moving plate. The points where the secondary magnetic field lines emanate are selected manually inside and nearby the eddy-current loops in order to guide the eye.

Appendix A. Eddy currents expressed through a stream function

By inserting $\phi(x, y, z)$ from (8) into (5), we find the components of electric current as:

$$j_x = \sigma \{-\partial_x \phi\} = -u\mu\sigma \left\{ \partial_x \partial_y \partial_n \left[\tanh^{-1} \frac{\Delta z}{R} \right] + \partial_x \phi_\infty \right\}, \quad (\text{A.1})$$

$$j_y = \sigma \{-\partial_y \phi - u B_z\} = u\mu\sigma \left\{ \partial_x \partial_x \partial_n \left[\tanh^{-1} \frac{\Delta z}{R} \right] - \partial_y \phi_\infty \right\}, \quad (\text{A.2})$$

$$j_z = 0, \quad (\text{A.3})$$

To simplify the above expression for j_y , we used B_z from (3) and the fact that

$$\partial_{xx} \left[\tanh^{-1} \frac{\Delta z}{R} \right] = -(\partial_{yy} + \partial_{zz}) \left[\tanh^{-1} \frac{\Delta z}{R} \right]. \quad (\text{A.4})$$

The obtained eddy currents are purely horizontal. Due to their two-dimensionality they can be expressed as

$$\mathbf{j} = \nabla \times (\psi \mathbf{e}_z)$$

where the stream function ψ is

$$\psi(x, y, z) = -u\mu\sigma \left\{ \partial_x \partial_n \left[\tanh^{-1} \frac{\Delta z}{R} \right] + \psi_\infty(x, y) \right\}, \quad (\text{A.5})$$

$$\psi_\infty(x, y) = -\lim_{z \rightarrow \infty} \partial_x \partial_n \left[\tanh^{-1} \frac{\Delta z}{R} \right] = \text{sgn}(\Delta z) \partial_x \partial_n \ln r.$$

The function $\psi_\infty(x, y)$ is a conjecture of the function $\phi_\infty(x, y)$ and it is needed to nullify eddy currents at the infinite distance from the dipole.

When the dipole is oriented arbitrarily, the general expression for ψ can be given explicitly with the aid of the tensor $\{\boldsymbol{\psi}^{(n)}\} = (\psi^{(x)}, \psi^{(y)}, \psi^{(z)})$ as follows:

$$\psi = u\mu\sigma (\mathbf{m} \cdot \{\boldsymbol{\psi}^{(n)}\}^T), \quad \{\boldsymbol{\psi}^{(n)}\} = \begin{pmatrix} F_0 + F_2 f_2 & F_2 g_2 & F_1 f_1 \end{pmatrix}, \quad (\text{A.6})$$

where

$$f_1 = \frac{\Delta x}{r}, \quad f_k = f_1 f_{k-1} - g_1 g_{k-1}, \quad (\text{A.7})$$

$$g_1 = \frac{\Delta y}{r}, \quad g_k = g_1 f_{k-1} + f_1 f_{k-1}; \quad (\text{A.8})$$

and

$$F_0 = -\frac{\Delta z}{2R^3}, \quad F_1 = \frac{r}{R^3}, \quad F_2 = -\frac{\Delta z}{2R^3} + \frac{\text{sgn}(\Delta z)}{r^2} \left[1 - \frac{|\Delta z|}{R} \right]. \quad (\text{A.9})$$

The eddy currents $\mathbf{j} = u\mu\sigma (\mathbf{m} \cdot \{\mathbf{j}^{(n)}\}^T)$ can be represented with the aid of the tensor $\{\mathbf{j}^{(n)}\}$ which is:

$$\{\mathbf{j}^{(n)}\} = \begin{pmatrix} C_1 g_1 + C_3 g_3 & C_1 f_1 - C_3 f_3 & C_2 g_2 \\ -3C_1 f_1 - C_3 f_3 & -C_1 g_1 - C_3 g_3 & C_0 - C_2 f_2 \\ 0 & 0 & 0 \end{pmatrix}, \quad (\text{A.10})$$

where

$$C_0 = \frac{3r^2}{2R^5} - \frac{1}{R^3}, \quad C_1 = \text{sgn}(\Delta z) \frac{3r|\Delta z|}{4R^5}, \quad C_2 = -\frac{3r^2}{2R^5}, \quad (\text{A.11})$$

$$C_3 = \text{sgn}(\Delta z) \left\{ \frac{|\Delta z|}{r^3} \left[\frac{3r}{4R^2} + \frac{1}{r} \right] + \frac{2}{r^3} \left[\frac{|\Delta z|}{R} - 1 \right] \right\}. \quad (\text{A.12})$$

The coefficients C_k tend to zero when $|\Delta z|$ goes to infinity. To find the eddy currents under the dipole, $r = 0$, one has to make a series expansion around $r = 0$. It gives:

$$C_0|_{r \rightarrow 0} = \frac{1}{|\Delta z|^3} \left\{ -1 + \frac{3r^2}{|\Delta z|^2} \right\} + O(r^4), \quad (\text{A.13})$$

$$C_1|_{r \rightarrow 0} = \frac{\text{sgn}(\Delta z) 3r}{4|\Delta z|^4} \left\{ 1 - \frac{5r^2}{2|\Delta z|^2} \right\} + O(r^5), \quad (\text{A.14})$$

$$C_2|_{r \rightarrow 0} = \frac{3r^2}{2|\Delta z|^5} \left\{ -1 + \frac{5r^2}{2|\Delta z|^2} \right\} + O(r^5), \quad (\text{A.15})$$

$$C_3|_{r \rightarrow 0} = \left\{ -\frac{\text{sgn}(\Delta z) 5r^3}{8|\Delta z|^6} \right\} + O(r^5) \quad (\text{A.16})$$

Thus at $r = 0$, C_0 is not zero. It is negative, hence, under the dipole, the direction of the eddy current is opposite to the y -axis, when the dipole is oriented along the z -direction in agreement with the right-hand rule. These series expansions results will also be useful afterwards in the analysis of the secondary magnetic field and forces acting upon the dipole.

It is important to notice that the functions f_k and g_k above depend on x and y only and do not depend on z . This will be useful below when we compute secondary magnetic field by integrating over the plate thickness $z_d \leq \hat{z} \leq z_u$. Moreover, these functions f_k and g_k are specially determined in such a way, that if the origin of the Cartesian coordinates is selected on the line passing through the magnetic dipole perpendicular to the plate, $x_0 = y_0 = 0$, then in the cylindrical coordinates, $\tan \varphi = y/x$, $f_k = \cos k\varphi$ and $g_k = \sin k\varphi$, i.e. the formulae above represent Fourier decomposition. There must be no angular dependence in a series expansion around $r = 0$ precisely at the point $r = 0$, therefore, C_0 is the only non vanishing coefficient in (A.13).

Appendix B. Primitive function for a thin layer

Here, we prove that the solution of (15) for an infinitely thin layer is given by:

$$A_0(\hat{z}) = \frac{w}{2} \left\{ \tanh^{-1} \frac{|\hat{\Delta}z(\hat{z})|}{\hat{R}(\hat{z})} + \ln r \right\}, \quad (\text{B.1})$$

where

$$\hat{R}(\hat{z})^2 = r^2 + |\hat{\Delta}z(\hat{z})|^2, \quad |\hat{\Delta}z(\hat{z})| = |\hat{z} - z_0| + |\hat{z} - z|.$$

Function $|\hat{\Delta}z(\hat{z})|$ can be understood from the following geometric interpretation. It is the sum of two terms, the first term, $|\hat{z} - z_0|$, is a distance between z_0 and \hat{z} , and the second term $|\hat{z} - z|$ is a distance between \hat{z} and z . Hence, the layer located at $z = \hat{z}$

subdivides the whole space into two half-spaces, $z > \hat{z}$ and $z < \hat{z}$, therefore the function $\hat{\Delta}z(\hat{z})$ in straight brackets takes the following form:

$$|\hat{\Delta}z(\hat{z})| = \begin{cases} |z + z_0 - 2\hat{z}| & \text{if } (\hat{z} - z)(\hat{z} - z_0) > 0; \\ |z - z_0| = |\Delta z| & \text{if } (\hat{z} - z)(\hat{z} - z_0) < 0. \end{cases} \quad (\text{B.2})$$

The first line is applicable when z and z_0 are both in the same half-space, hence, $\hat{\Delta}z(\hat{z}) = z + z_0 - 2\hat{z}$ in this case; the second line is applicable when z and z_0 are in different half-spaces, hence, $\hat{\Delta}z(\hat{z}) = z - z_0$ in this case. Then, it is important that $|\hat{\Delta}z(\hat{z})|$ shows a discontinuity in the first derivative at $z = \hat{z}$, i. e.:

$$\frac{d}{dz} |\hat{\Delta}z(\hat{z})| = \begin{cases} +1 & \text{if } z > \hat{z}, \\ -1 & \text{if } z < \hat{z}, \end{cases} \quad \text{hence} \quad \frac{1}{2} \frac{d^2}{dz^2} |\hat{\Delta}z(\hat{z})| = \delta(z - \hat{z}).$$

Now it is easy to ascertain by means of straightforward calculations that $\Delta A_0 = 0$ everywhere except for the point at $z = \hat{z}$ while at this point the formula (15) is valid because:

$$\begin{aligned} \frac{\partial^2 A_0}{\partial x^2} + \frac{\partial^2 A_0}{\partial y^2} &= \frac{w}{2} \frac{|\hat{\Delta}z(\hat{z})|}{\hat{R}^3}, \\ \frac{\partial^2 A_0}{\partial z^2} &= \frac{w}{2} \left\{ -\frac{|\hat{\Delta}z(\hat{z})|}{\hat{R}^3} \left[\frac{d}{dz} |\hat{\Delta}z(\hat{z})| \right]^2 + \frac{1}{\hat{R}} \frac{d^2}{dz^2} |\hat{\Delta}z(\hat{z})| \right\}, \\ \Delta A_0 &= \frac{\partial^2 A_0}{\partial x^2} + \frac{\partial^2 A_0}{\partial y^2} + \frac{\partial^2 A_0}{\partial z^2} = w \delta(z - \hat{z}) \frac{1}{R}, \end{aligned}$$

where the equality $\hat{R} = R$ at $z = \hat{z}$ is taken into account.

Appendix C. Tensor for the secondary field of a thin layer

As follows from the mapping (18-20) between induced field and eddy currents, and the tensor (A.10) for the eddy currents, the tensor of the induced field can be written down as:

$$\{\mathbf{b}^{(n)}\} = \begin{pmatrix} -3f_1 P_1 - f_3 P_3 & -g_1 P_1 - g_3 P_3 & f_2 P_2 - P_0 \\ -g_1 P_1 - g_3 P_3 & f_3 P_3 - f_1 P_1 & g_2 P_2 \\ P_0 - f_2 P_2 & -g_2 P_2 & -4f_1 P_1 \end{pmatrix} \quad (\text{C.1})$$

where the coefficients $P_k \equiv w P_{0,k}$ for an infinitely thin layer are obtained from the mapping $P_{0,k} \leftrightarrow \text{sgn}(\hat{\Delta}z) C_k / 2$, $k = 1, \dots, 3$, by keeping in mind that \hat{R} and $\hat{\Delta}z$ are to be used instead of R and Δz . Specifically,

$$\begin{aligned} P_{0,0} &= \text{sgn}(\hat{\Delta}z) \left\{ \frac{3r^2}{4\hat{R}^2} - \frac{1}{2\hat{R}^3} \right\}, \quad P_{0,1} = \frac{3r|\hat{\Delta}z|}{8\hat{R}^5}, \\ P_{0,2} &= -\text{sgn}(\hat{\Delta}z) \frac{3r^2}{4\hat{R}^5}, \quad P_{0,3} = \frac{|\hat{\Delta}z|}{2r^3} \left[\frac{3r}{4\hat{R}^2} + \frac{1}{r} \right] + \frac{1}{r^3} \left[\frac{|\hat{\Delta}z|}{\hat{R}} - 1 \right]. \end{aligned} \quad (\text{C.2})$$

The series expansion (A.13)-(A.16) is also applicable for (C.2). One can see that at $r = 0$ $P_{0,0}$ is not vanishing only, therefore, as follows from (C.1), the nonzero tensor components at $r = 0$ are $b_x^{(z)} = -b_z^{(x)}$ only.

Appendix D. Secondary magnetic field tensor for a half-space and for a layer with finite thickness

For a half-space and for a layer with finite thickness, the secondary field tensor is of the same structure as for an infinitely thin layer, Eq. (C.1), with the following coefficients for the half space:

$$\begin{aligned} P_{\infty,0} &= -\frac{\hat{\Delta}z}{8\hat{R}^3}, & P_{\infty,2} &= \text{sgn}(\hat{\Delta}z) \left\{ \frac{|\hat{\Delta}z|}{8\hat{R}^3} + \frac{1}{4r^2} \left[\frac{|\hat{\Delta}z|}{\hat{R}} - 1 \right] \right\}, \\ P_{\infty,1} &= \frac{r}{16\hat{R}^3}, & P_{\infty,3} &= \frac{r}{16\hat{R}^3} + \frac{1}{4\hat{R}r} + \frac{\text{sgn}(\hat{\Delta}z)\hat{R}}{2r^3} \left[\frac{|\hat{\Delta}z|}{\hat{R}} - 1 \right], \end{aligned} \quad (\text{D.1})$$

where $\hat{\Delta}z$ and \hat{R} should be computed at z_u .

For the layer with finite thickness, the tensor coefficients are:

$$P_k(z_u, z_d) = P_{\infty,k}(z_u) - P_{\infty,k}(z_d), \quad k = 0 \dots 4; \quad (\text{D.2})$$

where $P_{\infty,k}(z_u)$ and $P_{\infty,k}(z_d)$ are computed according to (D.1) with $\hat{\Delta}z$ and \hat{R} taken at points z_u and z_d .

Appendix E. Force field tensor

The force field tensor, $\{\mathbf{f}_i^{(n)}\} = \partial_i\{\mathbf{b}^{(n)}\}$ can be written down in the following way:

$$\{\mathbf{f}_x^{(n)}\} = \begin{pmatrix} 3Q_0 + 2f_2Q_2 + f_4Q_4 & g_2Q_2 + g_4Q_4 & 3f_1Q_1 + f_3Q_3 \\ g_2Q_2 + g_4Q_4 & Q_0 - f_4Q_4 & g_1Q_1 + g_3Q_3 \\ -3f_1Q_1 - f_3Q_3 & -g_1Q_1 - g_3Q_3 & 4Q_0 + 2f_2Q_2 \end{pmatrix} \quad (\text{E.1})$$

$$\{\mathbf{f}_y^{(n)}\} = \begin{pmatrix} g_2Q_2 + g_4Q_4 & Q_0 - f_4Q_4 & g_1Q_1 + g_3Q_3 \\ Q_0 - f_4Q_4 & g_2Q_2 - g_4Q_4 & f_1Q_1 - f_3Q_3 \\ -g_1Q_1 - g_3Q_3 & f_3Q_3 - f_1Q_1 & 2g_2Q_2 \end{pmatrix}, \quad (\text{E.2})$$

$$\{\mathbf{f}_z^{(n)}\} = \begin{pmatrix} -3f_1Q_1 - f_3Q_3 & -g_1Q_1 - g_3Q_3 & 4Q_0 + 2f_2Q_2 \\ -g_1Q_1 - g_3Q_3 & f_3Q_3 - f_1Q_1 & 2g_2Q_2 \\ -4Q_0 - 2f_2Q_2 & -2g_2Q_2 & -4f_1Q_1 \end{pmatrix}. \quad (\text{E.3})$$

The coefficients $Q_k \equiv w Q_{0,k}$ for a infinitely thin layer are:

$$\begin{aligned} Q_{0,0} &= \frac{3|\hat{\Delta}z|}{8\hat{R}^5} \left\{ \frac{5r^2}{2\hat{R}^2} - 1 \right\}, & Q_{0,1} &= \text{sgn}(\hat{\Delta}z) \frac{3r}{2\hat{R}^5} \left\{ \frac{5r^2}{4\hat{R}^2} - 1 \right\}, \\ Q_{0,2} &= \frac{15r^2|\hat{\Delta}z|}{8\hat{R}^7}, & Q_{0,3} &= \text{sgn}(\hat{\Delta}z) \frac{15r^3}{8\hat{R}^7}, \\ Q_{0,4} &= \frac{3|\hat{\Delta}z|}{2\hat{R}^3} \left[\frac{5r^2}{8\hat{R}^4} + \frac{3}{4\hat{R}^2} + \frac{1}{r^2} \right] + \frac{3}{r^4} \left[\frac{|\hat{\Delta}z|}{\hat{R}} - 1 \right]. \end{aligned} \quad (\text{E.4})$$

For a half-space, the coefficients $Q_k \equiv Q_{\infty,k}$ are obtained by using the same method as done before for the coefficients $P_{\infty,k}$. This involves an integration over the thickness of

the plate, $Q_{\infty,k} = \int_{-\infty}^{z_u} Q_{0,k}(\hat{z}) d\hat{z}$, and the results of the integration are:

$$Q_{\infty,0} = \frac{1}{16\hat{R}^3} \left\{ \frac{3r^2}{2\hat{R}^2} - 1 \right\}, \quad (\text{E.5})$$

$$Q_{\infty,1} = -\text{sgn}(\hat{\Delta}z) \frac{3r|\hat{\Delta}z|}{16\hat{R}^5}, \quad Q_{\infty,2} = \frac{3r^2}{16\hat{R}^5}, \quad (\text{E.6})$$

$$Q_{\infty,3} = -\text{sgn}(\hat{\Delta}z) \left\{ \frac{|\hat{\Delta}z|}{4r\hat{R}^3} \left[1 + \frac{3r^2}{4\hat{R}^2} \right] + \frac{1}{2r^3} \left[\frac{|\hat{\Delta}z|}{\hat{R}} - 1 \right] \right\}, \quad (\text{E.7})$$

$$Q_{\infty,4} = \frac{3}{4\hat{R}r^2} \left[1 + \frac{r^2}{4\hat{R}^2} + \frac{r^4}{8\hat{R}^4} \right] + \frac{3\hat{R}}{2r^4} \left[\frac{|\hat{\Delta}z|}{\hat{R}} - 1 \right]. \quad (\text{E.8})$$

For a plate with finite thickness we have:

$$Q_k = Q_{\infty,k}(z_u) - Q_{\infty,k}(z_d), \quad k = 0 \dots 4.$$

Appendix F. Non-zero tensor coefficients at $r = 0$ and $z = z_0$

The coefficients for a thin layer of thickness $w \ll |z_0 - \hat{z}|$ located at \hat{z} are:

$$P_0|_{r=0,z=z_0} \equiv w P_{0,0}|_{r=0,z=z_0} = -w \frac{\text{sgn}(z_0 - \hat{z})}{16|z_0 - \hat{z}|^3}, \quad (\text{F.1})$$

$$Q_0|_{r=0,z=z_0} \equiv w Q_{0,0}|_{r=0,z=z_0} = -w \frac{3}{128|z_0 - \hat{z}|^4}. \quad (\text{F.2})$$

The same coefficients for a half-space are obtained by integration over \hat{z} from z_u up to infinity

$$P_0|_{r=0,z=z_0} \equiv P_{\infty,0}|_{r=0,z=z_0} = -\frac{1}{32|z_0 - z_u|^2}, \quad (\text{F.3})$$

$$Q_0|_{r=0,z=z_0} \equiv Q_{\infty,0}|_{r=0,z=z_0} = -\frac{1}{128|z_0 - z_u|^3}. \quad (\text{F.4})$$

Finally, for a plate with finite thickness, $z_d \leq \hat{z} \leq z_u$:

$$P_0|_{r=0,z=z_0} = \frac{1}{32} \left\{ \frac{1}{|z_0 - z_d|^2} - \frac{1}{|z_0 - z_u|^2} \right\}, \quad (\text{F.5})$$

$$Q_0|_{r=0,z=z_0} = \frac{1}{128} \left\{ \frac{1}{|z_0 - z_d|^3} - \frac{1}{|z_0 - z_u|^3} \right\}. \quad (\text{F.6})$$

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